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DISPERSION AND BLOCKAGE EFFECTS IN THE FLOW OVER A SILL

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The problem of a homogeneous heavy liquid flow over a local obstacle is considered in the long-wave approximation. The steady and unsteady waves in the vicinity of the obstacle are described by second-order models of the shallow-water theory and their hyperbolic approximations. The flow in the vicinity of the leading and trailing edges of bluff bodies (sills and steps) is studied. The solution of the problem of the blocked zone upstream of the step is constructed.

Key words: *homogeneous liquid, equations of the shallow-water theory, dispersion effects, flow over a sill.*

INTRODUCTION

The second-order approximation of the shallow water theory is used to model interaction of nonlinear waves. There are various versions of equations to take into account the influence of dispersion effects on the structure of nonlinear surface and internal waves in the long-wave approximation [1–5]. Alternative formulations of the models have been recently proposed to describe these effects in flows of a homogeneous liquid with a free boundary. Such models describe the evolution of nonlinear dispersion waves within the framework of hyperbolic systems of equations of the shallow water theory [4]. The effect of non-hydrostatic distributions of pressure is taken into account in these models by using additional "internal" variables in the equations. A preliminary analysis shows that hyperbolic dispersion models can be used to describe the evolution of surface waves above an uneven bottom, in addition to more widespread non-hyperbolic models corresponding to the second-order approximation of the shallow water theory. The main advantage of dispersive hyperbolic equations of the shallow water theory is substantial simplification of the algorithms of numerical calculation of multidimensional unsteady flows and of formulation of the boundary conditions, in particular, near the coast line.

Another class of problems for which dispersive hyperbolic equations provide a more explicit formulation of the problem than models of the second-order approximation is associated with formulation of conditions for upstream control of the flow through insertion of a local obstacle into the channel. Conditions of this type in open-channel hydraulics were determined for the classical shallow water equations. The local obstacle controls the flow if a steady-state transcritical flow is formed in the vicinity of the obstacle. For flows in a horizontal channel, the condition of providing a critical flow above the wave crest determines the relation between the mass flow rate and the flow depth immediately ahead of the obstacle and is independent of the shape of the latter. For the second-order approximation, the condition of the critical flow is replaced by more complicated conditions of existence of a regular steady-state flow in the vicinity of a local constriction of the channel and do not allow the corresponding generalization in analyzing unsteady problems. Therefore, a hyperbolic analog of equations of the second-order approximation of the shallow water theory can be effectively used to formulate the boundary conditions in the problem of a liquid flow in a finite-length channel.

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Steady-state and unsteady problems of a plane-parallel flow of a homogeneous heavy liquid above a local obstacle are considered in the present paper. Models developed by Green and Naghdi [2, 3], by Serre [6], and hyperbolic approximations of these models are used to analyze the wave structure near the obstacle. The flows in the vicinity of the leading and trailing edges of bluff bodies (sill and step) are studied; the solution of the problem of a blocked zone upstream of a step is constructed.

1. MATHEMATICAL MODELS

1.1. Dispersive Hyperbolic Models. The following system of equations is used to describe nonlinear dispersion waves in continuous media [1]:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,$$
$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+P)}{\partial x}=0. \qquad (1)$$

Here ρ is the medium density and u is the mean velocity.

In contrast to gas-dynamic equations, which describe barotropic flows, the pressure P is assumed to depend not only on density but also on its material derivatives:

$$P=P\Big(\rho,\frac{d\rho}{dt},\frac{d^2\rho}{dt^2}\Big),\qquad \frac{d}{dt}=\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}. \qquad (2)$$

The class of equations considered includes the equations of bubbly liquids, equations of the shallow water theory, etc. [1–3]. These equations are used to describe media with "internal inertia," i.e., heterogeneous media where a certain part of the total energy transforms to the energy of small-scale motions, which produces a significant effect on the wave structure of the flows. Numerical implementation of one-dimensional models and, moreover, multidimensional analogs of Eqs. (1) and (2) involves some difficulties in imposing the boundary conditions (e.g., in the problem of waves rolling onto the shore), which are caused by non-hyperbolicity of the system considered.

These difficulties can be partly resolved by constructing the hyperbolic approximation (1). Systems of heterogeneous hyperbolic equations, which occupy an intermediate position between the first- and second-order approximations, were derived in [4] for the second-order approximation of the shallow water theory. A hyperbolic model is obtained by additional averaging of the equations of the second-order approximation and by introducing new "internal" variables. The scale of averaging is assumed to be rather small, which allows the values of the variables ρ and u to be replaced in the equations by their mean values. In calculating the total pressure, however, derivatives from the "instantaneous" values of density $\tilde\rho$ and velocity $\tilde u$ are used. As applied to Eq. (1), this means that the pressure depends on the averaged density ρ and material derivatives of $\tilde\rho$:

$$P=P\Big(\rho,\frac{d\tilde\rho}{dt},\frac{d^2\tilde\rho}{dt^2}\Big),\qquad \frac{d}{dt}=\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}.$$

The relation between the averaged and "internal" variables is found by expanding the functions $\tilde\rho(s,\xi)=\tilde\rho(s)$ into a Taylor series along the trajectory $x=x(s,\xi)$ (ξ is a fixed Lagrangian coordinate of the particle):

$$\tilde\rho(s)=\tilde\rho(t)+\tilde\rho'(t)(s-t)+\tilde\rho''(t)(s-t)^2/2+o(\tau^2),\qquad s\in(t-\tau,t+\tau). \qquad (3)$$

By virtue of Eq. (3), the mean value $\rho(t)$ is related to the instantaneous value $\tilde\rho(t)$ by the expression

$$\rho(t)=\frac{1}{2\tau}\int\limits_{t-\tau}^{t+\tau}\tilde\rho(s)\,ds=\tilde\rho(t)+\frac{1}{6}\,\tilde\rho''(t)\tau^2+o(\tau^2), \qquad (4)$$

which can be presented in the form

$$\tilde\rho''(t)=\alpha(\rho(t)-\tilde\rho(t))+O(\tau),\qquad \alpha=6/\tau^2. \qquad (5)$$

Using the main part in presentation (4) or (5), we obtain the averaged equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \qquad \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + \bar{P})}{\partial x} = 0,$$
$$\frac{\partial \tilde{\rho}}{\partial t} + u\,\frac{\partial \tilde{\rho}}{\partial x} = v, \qquad \frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} = \alpha(\rho - \tilde{\rho}). \tag{6}$$

Here $\bar{P}(\rho, \tilde{\rho}, v) = P(\rho, v, \alpha(\rho - \tilde{\rho}))$.

For a wide class of continuous media, Eqs. (6) form a heterogeneous hyperbolic system with two "sound" and two contact characteristics. As the parameter α increases (the averaging interval τ decreases), the solutions of Eqs. (6) approximate the solutions of the original system (1). The properties of dispersive hyperbolic systems are studied in more detail by an example of the second-order approximation of the equations of the shallow water theory.

1.2. Hyperbolic Approximation of the Green–Naghdi Equations. The evolution of long dispersion waves above an uneven bottom is described by various versions of the second-order approximation of the shallow water theory. In the present paper, we consider the Green–Naghdi equations [1, 2], which are the most suitable ones for calculating transcritical flows of a homogeneous heavy liquid in the vicinity of a local two-dimensional obstacle:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0,$$
$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\Big(hu^2 + \frac{1}{2}\,gh^2 + \frac{1}{3}\,h^2\,\frac{d^2h}{dt^2} + \frac{1}{2}\,h^2\,\frac{d^2z}{dt^2}\Big) + \Big(\frac{1}{2}\,\frac{d^2h}{dt^2} + \frac{d^2z}{dt^2} + g\Big)hz_x = 0, \tag{7}$$
$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}.$$

Here h is the depth, u is the flow velocity, $z = z(x)$ is the shape of the bottom, and g is the acceleration of gravity. A differential corollary of Eq. (7) for flows above a horizontal bottom ($z \equiv 0$) is the law of conservation of energy

$$\frac{\partial}{\partial t}\Big(h\Big(\frac{1}{2}\,u^2 + \frac{1}{2}\,gh + \frac{1}{6}\Big(\frac{dh}{dt}\Big)^2\Big)\Big) + \frac{\partial}{\partial x}\Big(hu\Big(\frac{1}{2}\,u^2 + gh + \frac{1}{6}\Big(\frac{dh}{dt}\Big)^2 + \frac{1}{3}\,h\frac{d^2h}{dt^2}\Big)\Big) = 0. \tag{8}$$

The hyperbolic approximation of Eqs. (7) and (8) implies replacement of the derivatives dh/dt and d^2h/dt^2 by the corresponding values for the instantaneous depth η and $d\eta/dt = v$ (see Sec. 1.1). The system of equations for the averaged variables h and u and "internal" variables η and v can be written in the form of the conservation laws:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \qquad \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\Big(hu^2 + \frac{1}{2}\,gh^2 + \frac{1}{3}\,\alpha h^2(h - \eta)\Big) = 0; \tag{9}$$

$$\frac{\partial}{\partial t}\Big(h\Big(\frac{1}{2}\,u^2 + \frac{1}{2}\,gh + \frac{1}{6}\,\alpha(h-\eta)^2 + \frac{1}{6}\,v^2\Big)\Big)$$
$$+ \frac{\partial}{\partial x}\Big(hu\Big(\frac{1}{2}\,u^2 + gh + \frac{1}{6}\,\alpha(h-\eta)^2 + \frac{1}{6}\,v^2 + \frac{1}{3}\,\alpha h(h-\eta)\Big)\Big) = 0,$$
$$\frac{\partial}{\partial t}(h\eta) + \frac{\partial}{\partial x}(hu\eta) = hv. \tag{10}$$

Note, by virtue of Eqs. (9) and Eqs. (10) can be replaced by the following non-divergent system of equations:

$$\frac{\partial \eta}{\partial t} + u\,\frac{\partial \eta}{\partial x} = v, \qquad \frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} = \alpha(h - \eta). \tag{11}$$

As systems (9), (10) and (9), (11) are equivalent on smooth solutions, hence, Eqs. (9) and (10) approximate the original system (7) as $\alpha \to \infty$.

The characteristics of system (9), (10) [or (9), (11)] are presented as follows:

$$\frac{dx}{dt} = \lambda^{\pm} = u \pm \sqrt{\Big(g + \alpha\Big(h - \frac{2}{3}\,\eta\Big)\Big)h}, \qquad \frac{dx}{dt} = \lambda_0 = u$$

(λ_0 is a multiple characteristic).

Note that the following condition has to be satisfied for approximation of Eq. (7):

$$|h-\eta| \ll h. \tag{12}$$

Therefore, Eqs. (9), (10) are hyperbolic-type equations in the equilibrium flow region ($h \equiv \eta$). In addition, the following should be noted: as for the second-order approximation of the shallow water theory, the approximate system is written ambiguously, because the mean depth h in the coefficients at the higher derivatives and their approximation can be replaced by the instantaneous depth η by virtue of Eq. (12). This replacement retains the long-wave asymptotics of solutions in the equilibrium flow region but alters the dispersion properties of the system. This circumstance is used below to simplify equations in the problem of a homogeneous liquid flow over a local obstacle.

For flows above an uneven bottom, it is also necessary to approximate the derivative d^2z/dt^2, which can be presented in the following form for a motionless bottom $z = z(x)$:

$$\frac{d^2z}{dt^2} = \frac{du}{dt}\, z' + u^2 z''. \tag{13}$$

If we replace the velocity u and its derivative du/dt in Eq. (13) by instantaneous quantities w and θ, respectively, by analogy with the presentation of d^2h/dt^2 via the instantaneous quantities η and v, the system is closed by Eqs. (11). Then, we have

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} = \theta, \qquad \frac{\partial \theta}{\partial t} + u\,\frac{\partial \theta}{\partial x} = \alpha(u-w). \tag{14}$$

The resultant system is hyperbolic if the following conditions are satisfied:

$$|h-\eta| \ll h, \qquad |\theta z' + w^2 z''| \ll g. \tag{15}$$

As was noted above, the first condition is the condition of approximation of the solutions of the original system (7). The second restriction in (15) guarantees that the vertical acceleration of the liquid layer caused by the flow over the local obstacle is small; therefore, the second restriction is also natural for the equations of the second-order approximation of the shallow water theory. Nevertheless, the conditions of hyperbolicity (15) are usually violated in calculations of bluff bodies, which reduces the range of applicability of the model constructed.

The range of hyperbolicity of the approximate system (7) can be expanded by replacing the mean depth h by the instantaneous depth η in the coefficients of system (7) at the second derivatives d^2h/dt^2 and d^2z/dt^2. By virtue of Eq. (12), the order of approximation of the original system is retained.

The equations of conservation of mass and momentum take the form

$$\begin{gathered}
\frac{\partial h}{\partial t} + \frac{\partial\,(hu)}{\partial x} = 0,\\
\frac{\partial\,(hu)}{\partial t} + \frac{\partial}{\partial x}\Big(hu^2 + \frac{1}{2}\,gh^2 + \frac{1}{3}\,\alpha\eta^2(h-\eta) + \frac{1}{2}\,\eta^2(\theta z' + w^2 z'')\Big)\\
+\Big(\frac{1}{2}\,\alpha(h-\eta) + \theta z' + w^2 z'' + g\Big)hz' = 0.
\end{gathered} \tag{16}$$

The closed system (11), (14), (16) is hyperbolic on an arbitrary solution. The characteristics of this system can be presented as follows:

$$\frac{dx}{dt} = \lambda^{\pm} = u \pm \sqrt{g\Big(h + \frac{\beta}{3}\,\frac{\eta^2}{h_0}\Big)}, \qquad \frac{dx}{dt} = u. \tag{17}$$

Here $\beta = \alpha h_0/g$ is a dimensionless parameter ($\beta \gg 1$). The parameter β and the characteristic depth h_0 depend on the chosen scale of averaging $\tau = \sqrt{6h_0/(\beta g)}$ for $\alpha = 6/\tau^2$. Thus, in contrast to the first- and second-order approximations of the shallow water theory, the hyperbolic system (11), (14), (16) contains an additional parameter β depending on the chosen scale of averaging of the original model. The solutions of Eqs. (11), (14), and (16) approximate the solutions of the Green–Naghdi equations as $\beta \to \infty$ and yield the solutions of the shallow water equations in the limit as $\beta \to 0$.

Note, as the parameter β changes from infinity to zero, system (11), (14), (16) provides a transition from smooth wave fronts typical of the second-order approximation of the shallow water theory to discontinuous waves arising in the classical shallow water theory. In particular, this system contains smooth soliton-like solutions, which describe the propagation of localized disturbances of the liquid with the free surface shape in the horizontal channel being retained. As the equilibrium model for the constructed dispersive hyperbolic model is the first approximation of the shallow water theory, smooth solitons exist for velocities of their propagation enclosed between the "equilibrium" velocity of propagation of long waves $\lambda_e^\pm = u \pm \sqrt{gh}$ and the "frozen" velocity $\lambda^\pm$ defined by Eq. (17) [5, Chapter 10]. Therefore, the structure of nonlinear wave fronts is smooth if the values of the parameter β are sufficiently high. The properties of the solutions for the hyperbolic approximation of the Green–Naghdi equations are considered in more detail in Sec. 2 by an example of the flow over a local obstacle.

2. STEADY-STATE FLOWS

We consider steady disturbances of a uniform flow ($h = h_0$ and $u = u_0 > 0$) over a symmetric local obstacle: $z(x-x_0) = z(x_0-x)$ and $z(x) = 0$ for $|x-x_0| > l$. The flow is assumed to be undisturbed at a rather large upstream distance from the obstacle ($x \to -\infty$). Depending on the original Froude number $\mathrm{Fr} = u_0/\sqrt{gh_0}$ (the values $\mathrm{Fr} > 1$ and $\mathrm{Fr} < 1$ refer to the supercritical and subcritical flows, respectively), different wave configurations can form in the vicinity of the obstacle. The classical equations of the shallow water theory allow mere determination of the flow regime (supercritical, subcritical, or transcritical flow) depending on the maximum height of the obstacle $z_{\max} = z(x_0)$. The flow regime is independent of the obstacle shape. Allowance for dispersion makes the wave pattern more versatile.

2.1. Supercritical Flow over the Obstacle. The steady-state flow equations for the Green–Naghdi model and its hyperbolic approximation can be partly integrated. By virtue of Eq. (8), the steady-state solutions of Eq. (7) are determined by the following integrals:

$$hu = Q = \mathrm{const},$$
$$(1/2)u^2 + g(h+z) + (1/6)u^2(2hh'' - (h')^2 + 3k((z')^2 + hz'')) = J = \mathrm{const}. \tag{18}$$

The value of the parameter $k = 1$ corresponds to the Green–Naghdi model (7), and the value $k = 0$ corresponds to the integrals of motion for the model [6]. Outside the obstacle ($z \equiv 0$), system (18) reduces to one first-order equation [7]

$$(h')^2 = \frac{3}{\mathrm{Fr}^2\, h_0^3}\,(h-h_0)^2(\mathrm{Fr}^2\, h_0 - h), \qquad |x - x_0| > l, \tag{19}$$

which can be integrated in quadratures. For $\mathrm{Fr} < 1$ and $x < x_0 - l$, there exists no flow different from a uniform flow ($h \equiv h_0$) with the above-indicated asymptotics as $x \to -\infty$. For $\mathrm{Fr} > 1$, the disturbance of the steady-state solution is a soliton whose shape is defined by the dependence

$$x = x_m \pm \int\limits_{h_m}^{h} \frac{\mathrm{Fr}\, h_0^{3/2}\, dh}{(h-h_0)\sqrt{3(\mathrm{Fr}^2\, h_0 - h)}}, \qquad h_m = \mathrm{Fr}^2\, h_0. \tag{20}$$

For $\mathrm{Fr} = 1$, Eq. (19) acquires the form

$$(h')^2 = 3(h_0 - h)^3/h_0^3. \tag{21}$$

The admissible solution of Eqs. (20) and (21) ($h \le h_0$) is monotonic. This solution is used below in the problem of a critical flow from a sill.

A supercritical flow ($\mathrm{Fr} > 1$) over a local obstacle can produce flows of two types, obtained by disturbing of a uniform flow and a solitary wave. As the flow outside the obstacle either is uniform ($h \equiv h_0$) or is part of a soliton, the function $h(x)$ at $|x - x_0| > l$ satisfies Eq. (19). By virtue of obstacle symmetry, we seek for a symmetric disturbed flow determined on the interval $x_0 - l < x < x_0$ by the solution of Eq. (18) with the following boundary conditions:

$$h'(x_0) = 0, \qquad h'(x_0 - l) = \Big(\frac{3(\mathrm{Fr}^2\, h_0 - h)}{\mathrm{Fr}^2\, h_0^3}\Big)^{1/2}(h - h_0). \tag{22}$$

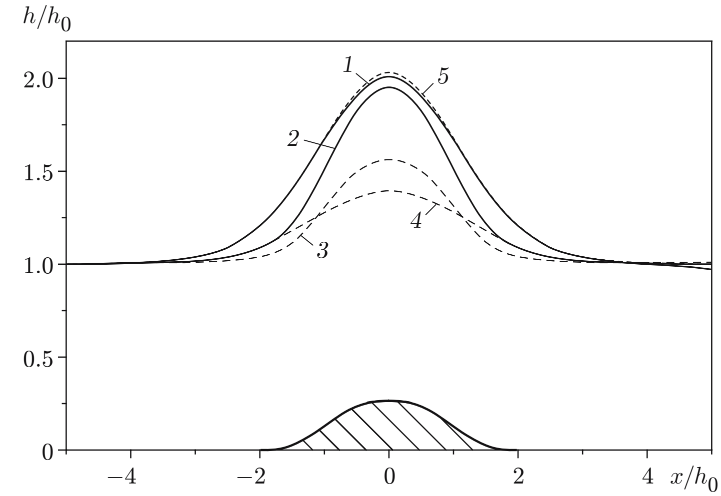

Fig. 1. Supercritical flow over an obstacle ($\mathrm{Fr} = 1.5$ and $z_{\max}/h_0 = 0.26$): soliton [Eqs. (18)] with $k = 1$ (curve 1) and 0 (curve 2), disturbances of a uniform flow [Eqs. (18)] with $k = 1$ (curve 3) and 0 (curve 4), and soliton [hyperbolic model (23)] (curve 5).

The solution of problem (18), (22) can be found numerically for the obstacle shape being defined. On the interval $x_0 < x < x_0 + l$, the solution is symmetric with respect to the obstacle center $x = x_0$. Figure 1 shows the solutions of the problem of a supercritical flow over the obstacle: $\mathrm{Fr} = 1.5$, $z = (1/2)z_{\max}[1 + (3/8)((\varepsilon - |x|)/\varepsilon)^5 - (5/4)((\varepsilon - |x|)/\varepsilon)^3 + (15/8)(\varepsilon - |x|)/\varepsilon]$, $|x| < l$, $\varepsilon = l/2$, $z_{\max} = 0.26h_0$, and $l = 2.62h_0$.

It should be noted that the solutions of Eqs. (18) for $k = 0$ and 1 are substantially different for a comparatively short obstacle shown in Fig. 1. As the obstacle length is increased (and its shape is retained unchanged), the curves come closer to each other, i.e., the simplified model (18) with $k = 0$ [6] can be used to model the flow over a rather smooth bulge on the bottom.

The steady-state symmetric solutions of the hyperbolic model satisfy the equations following from Eqs. (11), (14), and (16):

$$\eta' = \frac{v}{u}, \quad v' = \frac{\alpha(h-\eta)}{u}, \quad w' = \frac{\theta}{u}, \quad \theta' = \frac{\alpha(u-w)}{u},$$

$$h' = \frac{1}{\Delta}\Big(\Big(\frac{2\alpha}{3}\,\eta(h-\eta) - \frac{\alpha}{3}\,\eta^2 + \eta(\theta z' + w^2 z'')\Big)\eta' + \frac{1}{2}\,\eta^2\theta' z' + \eta^2 w z'' w' + \frac{1}{2}\,\eta^2\theta z'' + \frac{1}{2}\,\eta^2 w^2 z''' + hz'\Big(\frac{1}{2}\,\alpha(h-\eta) + \theta z' + w^2 z'' + g\Big)\Big), \tag{23}$$

$$hu = Q, \qquad \Delta = u^2 - gh - (1/3)\alpha\eta^2.$$

For flows above a horizontal bottom ($z \equiv 0$), Eqs. (23) are much simpler. The only dimensionless parameter determining the flow with the prescribed asymptotics ($h \to h_0$ and $u \to u_0$ as $x \to -\infty$) is the Froude number $\mathrm{Fr} = u_0/\sqrt{gh_0}$. With allowance for this asymptotics, the steady-state solutions of Eq. (16) satisfy the following system of equations:

$$hu = h_0u_0 = Q,$$
$$hu^2 + gh^2/2 + \alpha\eta^2(h-\eta)/3 = h_0u_0^2 + gh_0^2/2 = M, \qquad u(u\eta')' = \alpha(h-\eta). \tag{24}$$

The first two equations in system (24) yield the dependences $u = u(h)$ and $\eta = \eta(h)$. Note that

$$\frac{d\eta}{dh} = a(h) = \frac{gh + \alpha\eta^2/3 - Q^2/h^2}{\alpha\eta(\eta - 2h/3)} > 0$$

for $u^2 = Q^2/h^2 < gh + \alpha\eta^2/3$ and $h - \eta < \eta/2$. The first of these conditions, which shows that the steady-state flow is subcritical, is satisfied at rather large values of the parameter α (small times of averaging τ). The second condition is also consistent with the condition of approximation (12). Therefore, we can assume the implicit dependence $\eta = \eta(h)$ to be mutually unique.

The last equation in system (24), with allowance for the dependences $u = u(h)$ and $\eta = \eta(h)$ can be integrated:

$$\frac{1}{2} Q^2 \frac{a^2(h)}{h^2} (h')^2 = F(h) = \int\limits_{h_0}^{h} \alpha(s - \eta(s)) a(s)\, ds. \qquad (25)$$

The behavior of the function $F(h)$ in the neighborhood of the point h_0 determines the structure of the steady-state solution of system (24). As it follows from system (24) that $\eta(h_0) = h_0$, we obtain

$$\frac{dF}{dh}(h_0) = 0, \qquad \frac{d^2 F}{dh^2}(h_0) = \alpha a(h_0)(1 - a(h_0)).$$

Thus, the function $F(h)$ in the neighborhood of $h = h_0$ is positive if and only if $0 < a(h_0) < 1$ or

$$\sqrt{g h_0} < u_0 < \sqrt{g h_0 + \alpha h_0^2/3}. \qquad (26)$$

Condition (26) is a necessary condition for the existence of a smooth soliton-shaped solution of system (24). At

$$u_0 > \sqrt{g h_0 + \alpha h_0^2/3},$$

there are no smooth solutions that describe uniform flow disturbances, and an undular bore configuration is formed, which consists of the solution of the problem of the hydraulic jump and the adjacent periodic solution [4, Chapter 6]. We do not consider such a configuration in the present work; hence, conditions (26) are assumed to be satisfied. Note, for the supercritical flow (Fr > 1), conditions (26) are satisfied by choosing a rather large value of the parameter α in approximating Eq. (7).

In constructing the problem of a supercritical flow over a local symmetric obstacle within the framework of model (24), as for Eqs. (18), the flow outside the interval $(x_0 - l, x_0 + l)$ is assumed to be uniform ($h \equiv h_0$) or to be part of a soliton prescribed by Eq. (25). Therefore, we use the boundary conditions similar to Eqs. (22) to construct the flow above the frontal slope of the obstacle ($x_0 - l < x < x_0$):

$$h'(x_0) = 0, \qquad h'(x_0 - l) = h(2F(h))^{1/2}/(Qa(h)). \qquad (27)$$

The resultant solution on the interval $(x_0, x_0 + l)$ is reflected symmetrically: $h(x - x_0) = h(x_0 - x)$.

Curve 5 in Fig. 1 is the solution of problem (24), (27) with Fr $= 1.5$ and $\beta = 24$ corresponding to the disturbance of the soliton-shaped solution. With such a choice of the averaging scale [$\tau = (1/2)\sqrt{h_0/g}$], the solution obtained by the full model (18) and its hyperbolic approximation (24) almost coincide.

2.2. Transcritical Flows Over an Obstacle. As was noted above, it follows from Eq. (19) that there exist no steady-state subcritical flows (Fr < 1) other than a constant flow, which are disturbances of a uniform flow ($h \to h_0$ as $x \to -\infty$). Therefore, the steady-state solution within the framework of model (7) and its hyperbolic approximation (11), (14), (16) is disturbed only directly above the local obstacle [$h \equiv \text{const}$ for $x \in (-\infty, x_0 - l)$]. One of the most important problems of open-channel hydraulics is determining additional relations between the free-stream parameters h_0 and u_0, depending on the obstacle shape. The first-order approximation of the shallow water theory distinguishes only two steady-state regimes: subcritical and transcritical flows. In the first case, the obstacle produces only local disturbances in the uniform flow; in the second case, the flow becomes supercritical behind the obstacle. For transcritical flows, an additional condition that ensures controlling the upstream flow by means of the obstacle is the condition of having a critical flow above the obstacle crest ($z = z_{\max}$). For a given flow rate $Q = h_0 u_0$, therefore, the critical depth h_{cr} above the obstacle crest is determined as follows:

$$h_{\text{cr}} = (Q^2/g)^{1/3},$$

the free-stream Froude number being determined from the Bernoulli integral as a function of Q and $z_{\max}$. The obstacle shape has no effect on these relations. For rather short obstacles, however, the dispersion effects can substantially influence the wave structure in the vicinity of the obstacle and, as a consequence, the relations that describe the free-stream parameters. For the second-order approximation of the shallow water theory [e.g., for Eqs. (7)], the conditions of the critical flow are not formulated explicitly. Therefore, the admissible steady-state

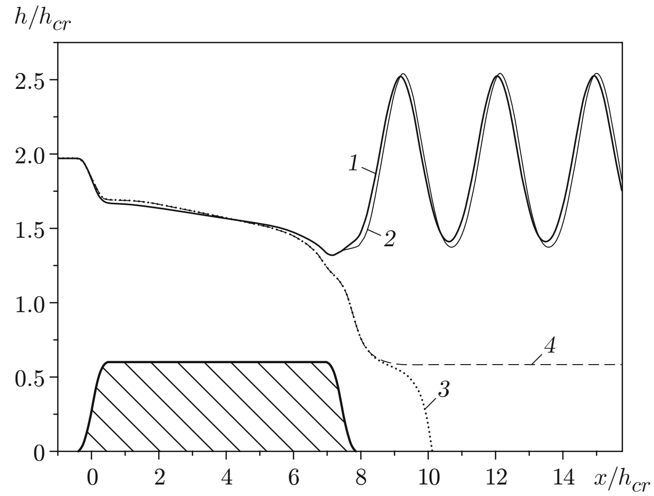

Fig. 2. Transcritical flow above a sill: solution of problem (18), (28) for the flow with leeward waves (curve 1) and with a "gradient catastrophe" (curve 3), hyperbolic model (23) for $\beta = 100$ (curve 2), and singular solution of Eq. (18) (curve 4).

solution is chosen through analyzing the behavior of the solutions [7, 8]. The flow on the interval $(-\infty, x_0 - l)$ is not disturbed, and the following initial conditions are imposed for Eq. (18):

$$h(x_0 - l) = h_0, \qquad h'(x_0 - l) = 0. \tag{28}$$

For a given flow rate Q and an obstacle in the form of a smoothed step whose shape is described by the relations

$$z(x) = \begin{cases} 0, & x < x_0 - \varepsilon, \\ \dfrac{1}{2}\, z_{\max}\Big(1 + \dfrac{3}{8}\Big(\dfrac{x - x_0}{\varepsilon}\Big)^5 - \dfrac{5}{4}\Big(\dfrac{x - x_0}{\varepsilon}\Big)^3 + \dfrac{15}{8}\,\dfrac{x - x_0}{\varepsilon}\Big), & x_0 - \varepsilon < x < x_0 + \varepsilon, \\ z_{\max}, & x_0 + \varepsilon < x < l, \end{cases}$$

various flow regimes can be formed, depending on the parameter h_0. Figure 2 shows the solutions of problem (18), (28) for close values of the parameter h_0, $z_{\max} = 0.6h_{\rm cr}$, and $l = 7.4h_{\rm cr}$. Changes in the value of the parameter h_0 by tenths fractions of a percent transforms the solution with leeward waves (curve 1) to a monotonic solution with unbounded derivatives (curve 3). Obviously, the solution corresponding to curve 3 cannot be realized, because the pressure at the channel bottom is $p \to -\infty$, and flow separation will occur. The singular (unstable) solution (curve 4), which describes a monotonic profile of the free surface in passing from a subscritical flow to a uniform supercritical flow, yields the necessary values of the parameter h_0, depending on the obstacle shape.

The hyperbolic model (23) can also be used to analyze the transcritical regime of the flow over the obstacle. Curve 2 in Fig. 2 shows the steady-state solution of Eqs. (23), which approximates the solution of problem (18), (28) for $\alpha = 100h_{\rm cr}/g$ ($\beta = 100$). It should be noted that small changes in the parameter h_0 for the hyperbolic model also lead to a "gradient catastrophe" in the solution, and there exists a "singular" solution separating these flow regimes.

2.3. Blockage of the Flow Over a Sill. In Sec. 2.2, we assumed that the flow is continuous above a local obstacle. In reals flows, there may be stagnant zones upstream and downstream of the obstacle, where the flow character is changed. In particular, a cavity can be formed behind the trailing edge of a rectangular sill. Within model (18), the transcritical flow above a rather wide sill ($z = z_{\max}$) is described by Eq. (21), because the uniform flow has to be critical far from the edge:

$$h = h_{\rm cr} = (Q^2/g)^{1/3}.$$

An additional condition consistent with the presence of a cavity behind the obstacle is the zero value of pressure on the channel bottom in the vicinity of the trailing edge ($x = x_1$):

$$p\Big|_{x=x_1} = \Big(gh + \frac{1}{2}\,\frac{Q^2}{h^2}\,(hh'' - (h')^2)\Big)\Big|_{x=x_1} = 0. \tag{29}$$

As Eqs. (18) above a smooth bottom ($z \equiv z_{\rm max}$) acquire the form

$$\frac{1}{2}\,\frac{Q^2}{h^2} + gh + \frac{1}{6}\,\frac{Q^2}{h^2}\,(2hh'' - h'^2) = \frac{3}{2}\,gh_{\rm cr}, \tag{30}$$

Eq. (21) with $h_0 = h_{\rm cr}$, and also Eqs. (29) and (30) yield the equation for $y_1 = h_1/h_{\rm cr}$ [the derivatives $h'(x_1)$ and $h''(x_1)$ are eliminated]:

$$y_1^3/3 + 3y_1 - 2 = 0.$$

The only solution of this equation on the interval $(0, 1)$ yields the dependence for the depth of the flow above the trailing edge:

$$h_1 \simeq 0.64Q^2/g.$$

By virtue of Eq. (21), the shape of the free surface in the vicinity of the edge can be found in quadratures.

A homogeneous liquid flow over a sill forms a stagnant zone upstream of the obstacle, which produces a significant effect on the flow character. Within the framework of the second-order approximation of the shallow water theory (18), the boundaries of the blocked zone can be found together with the shape of the free surface in the vicinity of the obstacle in the following formulation.

A steady-state flow with parameters h_0 and u_0 as $x \to -\infty$ encounters a local obstacle whose shape is defined by the function $z = z(x)$. A certain part of the boundary $z = z_0(x)$ corresponds to the obstacle shape on the segment of the attached flow. The other part of the boundary $z = z(x)$ is free $[z(x) \geq z_0(x)]$ and describes the region of the blocked liquid. We have to find the positions of the free boundaries $h = h(x)$ and $z = z(x)$ from Eq. (18) and the condition of continuity of the pressure p on the boundary of the blocked zone:

$$p\Big|_{y=z(x)} = gh + (1/2)u^2(hh'' - (h')^2) + ku^2(hz'' - h'z') = p_b - gz. \tag{31}$$

Here $p_b \equiv$ const is the pressure on the bottom upstream of the obstacle $[z_0(x) = 0]$. It is assumed thereby that the liquid in the blocked zone is at rest, and the pressure follows the hydrostatic law.

We also assume that the unknown boundary $z = z(x)$ smoothly joins the prescribed shape of the bottom $z = z_0(x)$. For $k = 0$, the problem of constructing a blocked zone in this formulation was solved in [9]. For $k = 1$, we seek an unbounded region occupied by the blocked zone: $z(x) > 0$ for $-\infty < x < x_0 - l$ and $z(x) \to 0$ for $x \to -\infty$. As the incoming flow is uniform, we have $p_b = p_{-\infty} = gh_0$; by virtue of Eqs. (18) and (31), the system of equations for determining the unknown boundaries $h = h(x)$ and $z = z(x)$ normalized by the conditions $Q = 1$ and $g = 1$ acquires the form

$$\begin{gathered} h'' = 12Jh - 6h^2 - 6zh - 6/h - 6h_0h - ((h')^2 + 6(z')^2 + 6h'z')/h, \\ z'' = ((h')^2 + 4h'z' + 3(z')^2 + 3)/h + 4h_0h - 6Jh + 2h^2 + 2hz, \\ J = 1/(2h_0^2) + h_0. \end{gathered} \tag{32}$$

Linearizing Eqs. (32) on a uniform flow $h \equiv h_0$, $z \equiv 0$, we obtain a system for the main flow disturbances $\tilde{h}$ and $\tilde{z}$:

$$\tilde{h}'' = 6(2/h_0^2 - h_0)\tilde{h} - 6h_0\tilde{z}, \qquad \tilde{z}'' = (-6/h_0^2 + 2h_0)\tilde{h} + 2h_0\tilde{z}.$$

Based on this system, we find the asymptotic solution (32) for $x \to -\infty$:

$$h = h_0 + \tilde{h}, \quad z = \tilde{z}, \quad \tilde{h} = \hat{h}\,e^{\lambda x}, \quad \tilde{z} = \hat{z}\,e^{\lambda x} \quad (\hat{z} > 0),$$

$$\lambda = \Big(6/h_0^2 - 2h_0 + \sqrt{(6/h_0^2 - 2h_0)^2 + 12/h_0}\Big)^{1/2}.$$

Then, using the asymptotic solution, we numerically construct the solution of Eqs. (32) for $z(x) \geq 0$. Thus, the shape of the blocked zone is determined with accuracy to an arbitrary shift along the x axis with respect to the prescribed local obstacle $z = z_0(x)$. Therefore, for the blocked zone and the obstacle to join smoothly, we only have to find the value x_*: $z(x_*) = z_0(x_*)$ and $z'(x_*) = z_0'(x_*)$. For $x > x_*$, the depth of the flow $h(x)$ is found by solving system (18) with $z(x) \equiv z_0(x)$.

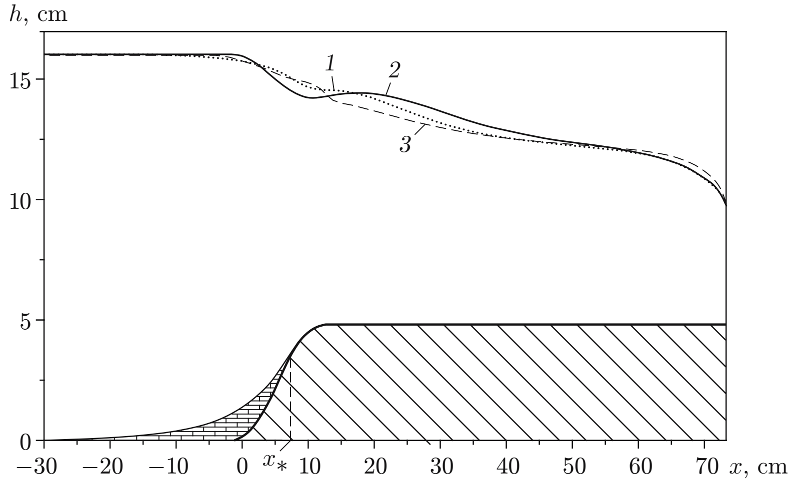

Fig. 3. Flow over a sill with a blocked zone ($\beta = 25$, $h_0 = 16$ cm, and $z_{\max} = 4.85$ cm): steady-state calculations by model (32) with a blocked zone (1), steady-state calculations without the blocked zone by model (18) (2) and by the dispersion model (23) (3).

Figure 3 shows the blocked zone calculated by model (32) (curve 1) and the corresponding regimes with continuous flows over a smooth sill by the Green–Naghdi model (curve 2) and by its hyperbolic approximation (23) (curve 3).

Note that the liquid is motionless in the blocked zone; therefore, the obstacle on the segment $(-\infty, x_*)$ can have an arbitrary shape $z_0(x)$ satisfying the inequality $z_0(x) \le z(x)$. Thus, the solution of problem (32) of the blocked zone with the above-mentioned asymptotics can be used for effective smoothing of bluff bodies with the help of the second-order approximation of equations of the shallow water theory. It is seen from Fig. 3 that the steady-state solution of the problem of a continuous flow over a smoothed sill (curves 2 and 3) is rather close to the steady-state solution of problem (32) of the blocked zone.

It should be noted that all three curves were obtained by using an additional condition (29) on the trailing edge of the sill. The validity of using this condition for selecting the steady-state solutions and determining the upstream conditions, based on the body shape, can be supported by passing to the unsteady problem. The simplest way to analyze the problem of an unsteady flow over a local obstacle with appropriate boundary conditions is to use the hyperbolic model (11), (14), (16). To determine the transcritical regime of the flow on the left boundary of the computational domain ($u > 0$), it is sufficient to set a constant flow rate and uniform flow conditions:

$$h = \eta, \quad w = u, \quad v = 0, \quad \theta = 0.$$

A supercritical flow is formed downstream of the local obstacle, and no conditions are imposed on the right boundary of the flow. The numerical solution of the problem of the flow over a local obstacle within the framework of the heterogeneous hyperbolic system (11), (14), (16) can be obtained by using standard algorithms. In the present work, we used Godunov's scheme to solve the problem of flow stabilization above the sill. For arbitrary initial data providing a transcritical regime above the obstacle, the flow becomes stabilized and approaches the steady flow (23) with an additional condition (24).

Figure 4 shows the results of the numerical solution of the problem of flow stabilization (curve 1), the corresponding solutions for the steady-state problem with conditions (24) obtained by the Green–Naghdi model (curve 2), and the hyperbolic approximation (9), (10) (curve 3).

The results of numerical calculations by the nonstationary model of the flow over a sill for different original flow rates are plotted in Fig. 5a. The value of the flow rate is determined by the critical depth $h_{\rm cr}$. The boundary conditions in the unsteady calculation were the same as in the calculation whose results are plotted in Fig. 4. The flow reaches the steady-state regime at $t > 20$ sec. Figure 5b shows the experimental results obtained in [10]. A comparison of Figs. 5a and 5b shows that the results calculated by the hyperbolic model offer an adequate description of the flow character.

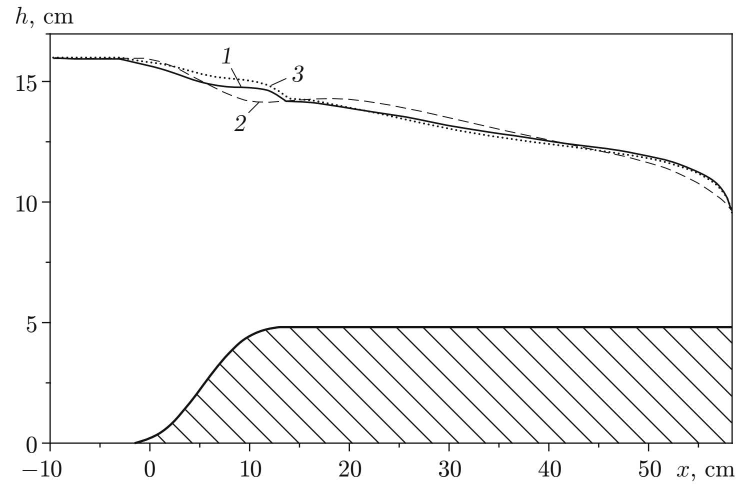

Fig. 4. Flow over a sill ($\beta = 25$, $h_0 = 16$ cm, and $z_{\max} = 4.85$ cm): curve 1 refers unsteady calculation by the hyperbolic model (11), (14), (16) curves 2 and 3 refer to steady calculations by the Green–Naghdi model (18) and hyperbolic model (23), respectively.

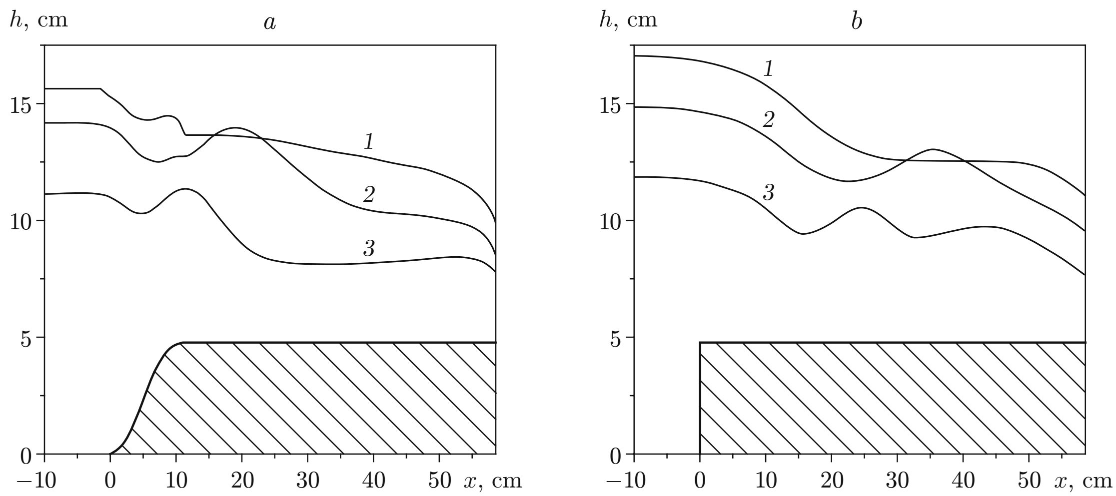

Fig. 5. Flow over a sill: (a) unsteady calculation by model (11), (14), (16); (b) experiment [10]; 1) $h_{\rm cr} = 8.1$ cm and $\beta = 25$; 2) $h_{\rm cr} = 6.43$ cm and $\beta = 50$; 3) $h_{\rm cr} = 4.28$ cm and $\beta = 25$.

An analysis of the results of calculating the steady-state and unsteady flows by the Green–Naghdi model and its hyperbolic approximation in Figs. 3–5 shows that the solution of the heterogeneous system of equations depends substantially on the obstacle smoothness. Discontinuities of the second derivative in the equation of the obstacle profile in the case of sill smoothing at the points where the smooth segment joins the horizontal surface are responsible for the typical inflections on the free surface above the conjugation points (see Figs. 3, 4, and 5a).

Introduction of a blocked zone into the flow pattern (curve 1 in Fig. 3) and reduction of the obstacle curvature in the case of a continuous flow eliminate this defect of the models. In a transcritical flow of a real liquid above a bluff body, such as a sill, the blocked zone is formed not only upstream of the obstacle but also above its leading edge (see Fig. 3 in [10]). This circumstance alters the effective shape of the obstacle and affects the formation of upstream flow parameters, which differ from those predicted by the models considered. This effect is manifested most clearly for comparatively short obstacles (curves 1 in Fig. 5).

CONCLUSIONS

The dispersive hyperbolic models (9) and (10) and model (11), (14), (16) are approximations of models corresponding to the second-order approximation of the shallow water theory and describe nonlinear dispersion effects in liquid flows with a free boundary. In contrast to models of the first- and second-order approximations of the shallow water theory, the model contains an explicit parameter α associated with the scale of averaging of the flow considered. These equations transform to the Green–Naghdi equations [2, 3] as $\alpha \to \infty$ ($\tau \to 0$) or to the classical equations of the shallow water theory as $\alpha \to 0$ ($\tau \to \infty$). The hyperbolic approximation of dispersion models is demonstrated to be rather effective if steady-state nonlinear waves generated by the local obstacle in the flow are described. In addition, the dispersive hyperbolic model simplifies the formulation and numerical implementation of unsteady problems of wave generation in finite-length channels.

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